

ON THE EXISTENCE OF STEEP TEMPORAL FUNCTIONS

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ABSTRACT. For Lipschitz continuous cone structures a new proof of the existence of Cauchy temporal functions is given using structure cycles.

1. INTRODUCTION

In Lorentzian geometry it is an important question, with a long history, to decide on good causality conditions when a spacetime (M, g) admits a Cauchy temporal function, i.e. a smooth function $\tau: M \rightarrow \mathbb{R}$ such that the differential does not vanish on the causal vectors and τ is onto \mathbb{R} along every inextendible causal curve. It is obvious that the existence of a Cauchy temporal function implies a splitting of the spacetime M into a product $\mathbb{R} \times N$ where the foliation $\{\{t\} \times N\}_{t \in \mathbb{R}}$ is spacelike. Further it implies that (i) the sets $J^+(p) \cap J^-(q)$ are compact and (ii) the spacetime is causal.

The question thus naturally arises whether this necessary condition is sufficient as well. The problem was solved with a positive answer in [2], [3] and [7] for smooth Lorentzian metrics. For continuous cone structures on the other hand [5] show the existence of a Cauchy temporal function under the assumption of (i) and (ii), the cone structure is stably causal. Their method though is very different from the one employed in Lorentzian geometry. An example given in [4] shows that the basic technique employed in Lorentzian geometry does not apply to the case of continuous cone structures.

Steep temporal function, i.e. functions τ with $|d\tau| \geq \sqrt{|g(v, v)|}$ for all causal vectors v , were used in the [8] for the isometric embedding problem of spacetime into Minkowski space. It was moreover shown in [8] that steep temporal functions always exist for globally hyperbolic spacetimes. Steep temporal functions are used to show existence of splittings with additional properties such as geometric bounds on the spacelike foliation, see [9].

The recent rise of Lorentz-Finsler geometry and questions arising in connection with metrics of low regularity, e.g. continuous metrics, repose the question of the existence of Cauchy and steep temporal function, respectively, in an extended context. It is the intend of these notes to give a proof of the existence of *L-steep temporal functions* for locally Lipschitz continuous cone structures (M, \mathcal{C}) under the assumptions (i) and (ii) above. Here a function $\tau: M \rightarrow \mathbb{R}$ is *L-steep* relative to a 1-homogenous continuous function $L: \mathcal{C} \rightarrow \mathbb{R}$ if $-d\tau|_{\mathcal{C}} \leq L$. As a corollary the existence of a *L-steep* temporal function for a every L implies the existence of a Cauchy temporal function. Simply consider a complete Riemannian metric h and $L_h := -\sqrt{h(v, v)}|_{\mathcal{C}}$. Then every L_h -steep temporal function is Cauchy.

The technique of proof relies on an adaptation of the proof of existence of a continuous time function, i.e. a function strictly monotonous along every causal curve, and an application of the theory of structures cycles as introduced in [12].

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2. THE RESULT

Consider a fiberwise convex cone field \mathcal{C} on a smooth manifold M i.e. a subset $\mathcal{C} \subseteq TM$ with $\mathbb{R}^+ \cdot \mathcal{C} = \mathcal{C}$, $\mathcal{C}_p := \mathcal{C} \cap TM_p$ convex for all $p \in M$ and $\pi(\mathcal{C}) = M$ where $\pi: TM \rightarrow M$ denotes the canonical projection. Recall that a cone \mathcal{K} in a vector space V has *compact base* if there exists $l \in V^*$ with $l|_{\mathcal{K} \setminus \{0\}} > 0$ and $l^{-1}(1) \cap \mathcal{K}$ is compact. The cone field \mathcal{C} is said to have compact base if $\mathcal{C}_p := \mathcal{C} \cap TM_p$ has compact base for all $p \in M$.

As in [5] a cone field with compact base is said to be *continuous* if the intersections with the unit balls of an arbitrary Riemannian metric are continuously depending on the base point with respect to the Hausdorff topology. Further a cone structure is *locally Lipschitz* if the intersections with the unit tangent bundle of a Riemannian metric are locally Lipschitz with respect to the Hausdorff distance.

Recall from [6] the definition of an absolutely continuous curves to be \mathcal{C} -causal which is phrased here in the more general context of continuous cone fields.

Definition 2.1. Let (M, \mathcal{C}) be continuous and I an interval.

- (i) An absolutely continuous curve $\gamma: I \rightarrow M$ is \mathcal{C} -causal (or simply *causal*), if $\dot{\gamma}(t) \in \mathcal{C}$ whenever $\dot{\gamma}(t)$ exists.
- (ii) A causal curve $\gamma: I \rightarrow M$ is *timelike* if for all $s \in I$ there exists $\varepsilon, \delta > 0$ such that for every $t \in I$ $\text{dist}(\dot{\gamma}(t), \partial\mathcal{C}) \geq \varepsilon|\dot{\gamma}(t)|$, whenever $\dot{\gamma}(t)$ exists and $|s - t| < \delta$.

Define $J^+(p)$ to be the set of points $q \in M$ such that there exists a causal curve with initial and terminal point p and q , respectively. $J^-(p)$ is defined with the roles of p and q exchanged as initial and terminal point. $I^+(p)$ and $I^-(p)$ have the same definition as $J^\pm(p)$ with causal replaced by timelike. For $U \subseteq M$ open define J_U^\pm and I_U^\pm as before for the restriction $(U, \mathcal{C}|_U)$.

A cone structure is said to be *causal* if it does not admit a closed causal curve.

Definition 2.2. A locally Lipschitz cone structure (M, \mathcal{C}) is *globally hyperbolic* if (M, \mathcal{C}) is causal and the sets $J^+(p) \cap J^-(q)$ are compact for all $p, q \in M$.

Let $L: \mathcal{C} \rightarrow \mathbb{R}$ be a locally Lipschitz continuous and 1-homogeneous function. A C^1 -function $\tau: M \rightarrow \mathbb{R}$ is said to be *L-steep* if $-d\tau(v) \leq L(v)$ for all $v \in \mathcal{C}$.

Theorem 2.3. For any globally hyperbolic cone structure $\mathcal{C} \subseteq TM$ and any 1-homogeneous locally Lipschitz function $L: \mathcal{C} \rightarrow \mathbb{R}$ there exists a *L-steep smooth function* $\tau: M \rightarrow \mathbb{R}$.

Recall from above that the theorem implies the existence of a Cauchy temporal function. The existence of a Cauchy temporal function induces a splitting $M \cong \mathbb{R} \times N$ such that TN_x is a supporting hyperplane of \mathcal{C} for all $(t, x) \in \mathbb{R} \times N$. Note that this result complements the known results on the existence of Cauchy temporal function for continuous cone fields in [5] and the existence of steep temporal functions for Lorentzian metrics in [8].

3. PROOF OF THE RESULT

Existence of a continuous time function. Let (M, \mathcal{C}) be a cone field. A continuous function $t: M \rightarrow \mathbb{R}$ is a *time function* if f is strictly increasing along every \mathcal{C} -causal curve. The first step in the proof of the theorem is the following statement.

Proposition 3.1. *Let (M, \mathcal{C}) be a globally hyperbolic cone structure. Then there exists a time function.*

As stated in [7] the property that $\partial I^\pm(p)$ are Lebesgue neglectable is the main point in the proof of existence of time functions for e.g. globally hyperbolic Lorentzian manifolds. The reasoning in the proof survives the change of setup to locally Lipschitz continuous cone structures. The reader is invited to check the details of this claim in [7].

The fact that the $\partial I^\pm(p)$ are Lebesgue neglectable follows from the observation that they are local Lipschitz hypersurfaces. In the Lorentzian case this is shown in [10], p. 413 ff. There the property is used that for Lorentzian manifolds the closure of $I^\pm(x)$ contains $J^\pm(x)$ for all $x \in M$.

Proposition 3.2. *Let (M, \mathcal{C}) be a locally Lipschitz cone structure. Then*

$$J^\pm(x) \subseteq \overline{I^\pm(x)}$$

for all $x \in M$.

With this proposition the fact that $\partial I^\pm(p)$ are local Lipschitz hypersurfaces is proven mutatis mutandis as in [10]. Note that the proposition fails in general without the assumption of Lipschitz continuity of the cone structure. For an example see [4] Example 1.11. It remains to prove Proposition 3.2.

The discussion starts with some basic facts on causal curves. The proof are omitted since the arguments are merely repetitions of their counterparts in Lorentzian geometry and therefore well known.

Remark 3.3. Note that for continuous cone fields every point p has an open neighborhood U admitting a smooth function $\tau: U \rightarrow \mathbb{R}$ and $\varepsilon > 0$ such that $d\tau(v) \geq \varepsilon|v|$ for all $v \in \mathcal{C}|_U$.

Lemma 3.4. *Let h be a Riemannian metric on M , $U \subseteq M$ be open, $\tau: U \rightarrow \mathbb{R}$ be a C^1 function such that $d\tau(v) > 0$ for all $v \in \mathcal{C}|_U$ and $K \subseteq U$ a compact set. Then there exists $\varepsilon > 0$ such that*

$$\tau(\eta(b)) - \tau(\eta(a)) \geq \varepsilon L^h(\eta)$$

for any \mathcal{C} -causal curve $\eta: [a, b] \rightarrow K$ where $L^h(\eta)$ denotes the length of η with respect to h .

Lemma 3.5. *Every point admits an arbitrary small open neighborhood U such that $J_U^+(x) \cap J_U^-(y)$ is compact for all $x, y \in U$.*

Proposition 3.6. *For an sufficiently small open neighborhood U as in Lemma 3.5 and any pair of points in U the set of causal paths in U connecting these points is compact up to parameterization in the C^0 -topology, i.e. for every causal curve one can choose a parameterization such that the set of causal curves with these parameterizations is compact.*

Proposition 3.7. *Let (M, \mathcal{C}) be a locally Lipschitz cone structure and $\gamma: [a, b] \rightarrow M$ a causal curve such that $\dot{\gamma}(t) \in \text{int } \mathcal{C}$ for one $t \in [a, b]$. Then γ can be deformed with fixed endpoints to a piecewise smooth timelike curve.*

Proof. Choose a complete Riemannian metric h on M and a timelike vector field X . Denote with Φ the flow of X and consider the map $F: [a, b] \times \mathbb{R} \rightarrow M$, $(s, t) \mapsto \Phi_t(\gamma(s))$. In order for X to be complete one can assume that $h(X, X) = 1$. For parameters $s \in [a, b]$ where $\dot{\gamma}(s)$ exists F is differentiable at (s, t) for all $t \in \mathbb{R}$. If $\dot{\gamma}(s) \parallel X_{\gamma(s)}$ whenever $\dot{\gamma}$ exists then γ is a reparameterization of a flow line of X . Deforming the parameterization to a smooth one yields the desired deformation.

Further if $\dot{\gamma}(s) \nparallel X_{\gamma(s)}$ the $dF(s, t)$ has rank 2 for all t . For these points the image of dF is a plane intersecting the interior of \mathcal{C} . Therefore there exists a unique $f(s, t)$ such that $dF(1, f(s, t)) \in \partial\mathcal{C}$. For the remainder of the argument one caps f at -1 , i.e. consider $\max\{f, -1\}$ instead of f . Extend f by 0 to the remaining points. Note that f is Lebesgue measurable and for every s the restriction of f to $\{s\} \times \mathbb{R}$ is Lipschitz with a Lipschitz constant $\text{Lip}(s)$. $\text{Lip}(s)$ is bounded by the local Lipschitz constant of \mathcal{C} . Therefore the map $s \mapsto \text{Lip}(s)$ is essentially bounded.

Lemma. *Let $f: [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ be a bounded Lebesgue measurable function such that for every $s \in [0, 1]$ the function $t \mapsto f(s, t)$ is $\text{Lip}(s)$ -Lipschitz with $\text{Lip} \in L^1([a, b], \mathbb{R})$. Then for every $t_0 \in \mathbb{R}$ there exists a unique almost everywhere solution to the equation*

$$\begin{cases} t'(s) &= f(s, t(s)), \\ t(0) &= t_0 \end{cases}$$

Proof. The proof is the same fixed point argument as in the case of the classical Theorem of Picard and Lindelöf. Choose $\varepsilon_s > 0$ for every $s \in [0, 1]$ such that

$$\int_{s-\varepsilon_s}^{s+\varepsilon_s} \text{Lip}(\sigma) d\sigma < 1.$$

This induces an open covering of $[0, 1]$ and there exists a finite subcovering $\{[a_i, b_i]\}$ with $a_{i+1} < b_i$. So if one proves existence and uniqueness of the solution on every interval $[a_i, b_i]$ the local solution join to one unique solution on $[0, 1]$. Hence it suffices to consider the case that $\int_0^1 \text{Lip}(\sigma) d\sigma \leq \lambda < 1$.

For given $t_0 \in \mathbb{R}$ consider the operator

$$\Psi: C^0([0, 1]) \rightarrow C^0([0, 1]), \quad y \mapsto \left[s \mapsto t_0 + \int_0^s f(\sigma, y(\sigma)) d\sigma \right].$$

For $y, z \in C^0([0, 1])$ and $s \in [0, 1]$ one then has

$$\begin{aligned} |\Psi(y)(s) - \Psi(z)(s)| &\leq \int_0^s |f(\sigma, y(\sigma)) - f(\sigma, z(\sigma))| d\sigma \leq \int_0^s \text{Lip}(\sigma) |y(\sigma) - z(\sigma)| d\sigma \\ &\leq \|y - z\|_{C^0} \int_0^s \text{Lip}(\sigma) d\sigma \leq \lambda \|y - z\|_{C^0}. \end{aligned}$$

The claim follows now from the Banach fixed point Theorem. \square

The lemma shows that the measurable distribution defined by $(1, f(s, t))$ is uniquely integrable. So if $\dot{\gamma}$ is somewhere timelike the curve $s \mapsto (s, 0)$ can be deformed in $[a, b] \times \mathbb{R}$ with fixed endpoints to a smooth curve η uniformly transversal to the distribution $\langle (1, f(s, t)) \rangle$, i.e. there exists $\delta > 0$ such that

$$|\dot{\eta} - (1, f)| \geq \delta |\dot{\eta}|.$$

The curve $F \circ \eta$ is Lipschitz and timelike. This curve can now be deformed with fixed endpoints to a smooth timelike curve, e.g. via mollification. \square

Proof of Proposition 3.2. Let $q \in J^+(p)$. Choose a causal curve from p to q and a timelike curve $\gamma: [0, 1] \rightarrow M$ starting at q . The concatenation of both curves satisfies the hypotheses of the previous proposition. Therefore one has $\gamma(t) \subseteq I^+(p)$ for all $t > 0$. \square

Before proceeding to the proof of existence of smooth temporal functions this is a good place to remark on the definition of causal curves. The definition was given by an infinitesimal condition, i.e. a constraint on the tangent vectors of a curve. With the definition of J^+ though one can give the following more geometric and seemingly more general definition of a causal curve:

Definition 3.8. A continuous curve $\gamma: [a, b] \rightarrow M$ is causal iff for all $t \in [a, b]$ and all neighborhoods U of $\gamma(t)$ there exists a smaller neighborhood $V \subseteq U$ of $\gamma(t)$ such that for all $s_1 < s_2$ in the preimage of V there exists an absolutely continuous causal curve in V from $\gamma(s_1)$ to $\gamma(s_2)$.

Note that this definition is independent of the parameterization, whereas the previous one needed an absolutely continuous one. Nevertheless not much is gained according to the following lemma.

Lemma 3.9. *Every continuous causal curve admits a locally Lipschitz parameterization.*

Therefore Definition 3.8 does not yield any new geometric objects, i.e. no curves that were not causal before even up to the freedom to choose the parameterization.

Proof. Let $\gamma: [a, b] \rightarrow M$ be a continuous causal curve. Choose $a = t_0 < t_1 < \dots < t_n = b$ such that $\gamma([t_i, t_{i+1}])$ is contained in a neighborhood U_i admitting a smooth function τ_i as in Remark 3.3. Then $\tau_i \circ \gamma|_{[t_i, t_{i+1}]}$ is monotone increasing and if $\gamma(s_1) \neq \gamma(s_2)$ for $s_1 < s_2 \in [t_i, t_{i+1}]$ one has $\tau_i \circ \gamma(s_1) < \tau_i \circ \gamma(s_2)$. Now parameterizing $\gamma|_{[t_i, t_{i+1}]}$ to $\bar{\gamma}: [\bar{t}_i, \bar{t}_{i+1}] \rightarrow M$ with $\tau_i(\bar{\gamma}(s_2)) - \tau_i(\bar{\gamma}(s_1)) = s_2 - s_1$ for all $s_{1,2} \in [\bar{t}_i, \bar{t}_{i+1}]$ yields a Lipschitz parameterization of $\gamma|_{[t_i, t_{i+1}]}$. Joining the local parameterizations to consecutive pieces yields a local Lipschitz parameterization of γ . \square

Existence of a L -steep function. Consider on M the space of smooth and compactly supported 1-forms $\Gamma_c^\infty(T^*M)$ with the C^∞ -Fréchet topology. The dual space $\Gamma_c^\infty(T^*M)'$, i.e. the space of 1-currents, is a strong dual of $\Gamma_c^\infty(T^*M)$, i.e. $(\Gamma_c^\infty(T^*M))' \cong \Gamma_c^\infty(T^*M)$. Further bounded sets in $\Gamma_c^\infty(T^*M)$ as well as $\Gamma_c^\infty(T^*M)'$ are precompact, see [11]. Following [11] one calls a 1-currents a *Dirac current* if it is given by the evaluation of forms at a single tangent vector. For a cone structure (M, \mathcal{C}) the *cone of structure currents* \mathcal{C} is then defined as the closed cone generated by the Dirac currents supported at points in \mathcal{C} . For a compact subset K of M let \mathcal{C}_K be the structure cone generated by the restriction of \mathcal{C} to K . Recall from [12] that for a compact cone $\mathcal{K} \subseteq V$ and a linear functional $l \in V^*$ such that $l^{-1}(1) \cap \mathcal{K}$ is compact the set $l^{-1}(1) \cap \mathcal{K}$ is called a *base* of \mathcal{K} .

Proposition 3.10 (Proposition I.5 in [12]). *The cone of structure currents \mathcal{C}_K for every compact subset K of M is a cone with compact base.*

A closed current in \mathcal{C} is said to be a *structure cycle*.

Proposition 3.11. *Let (M, \mathcal{C}) be a cone structure such that for all compact $K \subset M$ \mathcal{C}_K only contains the trivial cycle. Then (M, \mathcal{C}) admits a smooth temporal function $\tau_0: M \rightarrow \mathbb{R}$, i.e. τ_0 is smooth with $d\tau_0|_{\mathcal{C} \setminus \overline{0}} > 0$.*

The statement of the proposition with M replaced throughout the claim by the compact set K is already contained in [12], Theorem III.1.

Lemma 3.12. *Let $K_1 \subset K_2$ be compact subsets of M such that \mathcal{C}_{K_2} contains only the trivial cycle. Let ω_1 be an exact 1-form on K_1 transverse to \mathcal{C} . Then there exists an exact extension ω_2 to K_2 transverse to \mathcal{C} .*

Proof. The argument is taken from Theorem I.7. in [12]. The currents on K_1 form a closed subspace of the space of currents on K_2 . Any base of \mathcal{C}_{K_2} does not intersect the closed space \mathcal{K}_2 spanned by the kernel of ω_1 and the structure cycles in K_2 by assumption. Since according to Proposition 3.10 the base of \mathcal{C}_{K_2} is compact and convex one can extend, using the Theorem of Hahn-Banach, \mathcal{K}_2 to a closed subspace of codimension one disjoint from any base of \mathcal{C}_{K_2} . The extension ω_2 is then given by declaring this space to be kernel of ω_2 . \square

Proof of Proposition 3.11. Choose a compact exhaustion $\{K_i\}_{i \in \mathbb{N}}$ of M with $K_i \subseteq K_{i+1}$. Then the direct limit ω of the exact forms ω_i on K_i constructed in Lemma 3.12 is exact and transverse to \mathcal{C} . Any primitive τ_0 of ω is the desired temporal function. \square

One says that the cone structure \mathcal{C}' is *strictly wider* than the cone structure \mathcal{C} if every \mathcal{C} -causal nonzero vector v is \mathcal{C}' -timelike, i.e. $v \in \text{int } \mathcal{C}'$. Following this notion of wideness one defines a cone structure to be *stably causal* on a set S if there exists a causal cone structure \mathcal{C}' strictly wider than \mathcal{C} on $\text{int } S$ and coinciding with \mathcal{C} on $\overline{M \setminus S}$. The cone structure (M, \mathcal{C}) is stably causal if it is stably causal on M . Note that [12] Theorem III.1 and the remark thereafter imply that stable causality implies the nonexistence of nontrivial structure cycles.

Proposition 3.13. *Let (M, \mathcal{C}) be globally hyperbolic. Then \mathcal{C} is stably causal on every compact subset of M .*

Proof. Let $f: M \rightarrow \mathbb{R}$ be a continuous \mathcal{C} -time function and $K \subseteq M$ be a compact set. Assume that the claim is false, i.e. for every $n \in \mathbb{N}$ exists a cone structure \mathcal{C}_n strictly wider than \mathcal{C} on $\text{int } K$, coinciding on $\overline{M \setminus K}$ and

$$\text{dist}_{\text{Hd}}(\mathcal{C}^1, \mathcal{C}_n^1) \rightarrow 0$$

for $n \rightarrow \infty$ such that \mathcal{C}_n admits a causal loop $\gamma_n: S^1 \rightarrow M$. It is clear that $\gamma_n \cap K \neq \emptyset$ since $\mathcal{C}_n|_{\overline{M \setminus K}} \equiv \mathcal{C}|_{\overline{M \setminus K}}$ is causal. Reparameterize the γ_n 's with respect to Riemannian arclength. If the Riemannian arclength stays bounded as $n \rightarrow \infty$ consider the sequence $\{\gamma_n\}_n$ itself in what follows. If the sequence of Riemannian arclength is not bounded dissect γ_n into pieces $\eta_{n,k}: [a_{n,k}, b_{n,k}] \rightarrow M$ of equal length in the interval $[1, 2]$. Since

$$\sum_k f(\eta_{n,k}(b_{n,k})) - f(\eta_{n,k}(a_{n,k})) = 0$$

there exists k_n with

$$f(\eta_{n,k_n}(b_{n,k_n})) - f(\eta_{n,k_n}(a_{n,k_n})) \leq 0.$$

Since f is a \mathcal{C} -time function one has $\eta_{n,k_n}([a_{n,k_n}, b_{n,k_n}]) \cap K \neq \emptyset$. Further by the Theorem of Arzela-Ascoli there exists a subsequence of $\{\eta_{n,k_n}\}_n$ converging uniformly to a \mathcal{C} -causal curve $\eta: [a, b] \rightarrow M$, cf. the proof of Proposition 3.6 or Lemma 4.1 in [5]. Note that

$$(1) \quad f(\eta(b)) - f(\eta(a)) \leq 0.$$

Applying Lemma 3.4 to a local temporal function τ around $\eta(a)$ and one of the cone structures \mathcal{C}_n one sees that η is not constant. Together with (1) this yields a contradiction. \square

Corollary 3.14. *If (M, \mathcal{C}) is globally hyperbolic it admits a temporal function.*

Define a new Lipschitz continuous cone structure \mathcal{D} on $M \times S^1$ by considering the convex hull of the graph of L and $\{0_p\} \times \mathbb{R}$ in every $TM_p \times \mathbb{R} \cong T(M \times S^1)_{(p, \theta)}$. Note that the cones \mathcal{D} still have compact base.

Proposition 3.15. *Let (M, \mathcal{C}) be as in Proposition 3.11. Then for every compact $K \subseteq M$ the restriction of \mathcal{D} to $K \times S^1$ does not support any nontrivial nullhomologous structure cycles.*

Proof. Consider the pullback $\pi^* \tau_0: M \times S^1 \rightarrow \mathbb{R}$ of τ_0 under the canonical projection $\pi: M \times S^1 \rightarrow M$. Then one has $d\pi^* \tau_0|_{\mathcal{D}} \geq 0$ with $d\pi^* \tau_0|_{\mathcal{D}}(v) = 0$ iff v is a non negative multiple of ∂_θ . Therefore any structure cycle must be supported in the cone structure $\text{pos}\{\partial_\theta\}$. By considering the closed 1-form $d\theta$ one immediately sees that only the trivial structure cycle is nullhomologous. \square

Lemma 3.16. *Let $K_1 \subseteq K_2$ be compact subsets of M such that \mathcal{C}_{K_2} contains only the trivial cycle. Then any closed form ω_1 on $K_1 \times S^1$ cohomologous to the restriction of $d\theta$ to $K_1 \times S^1$ and transverse to $\mathcal{D}|_{K_1 \times S^1}$ has an extension ω_2 cohomologous to the restriction of $d\theta$ to $K_2 \times S^1$ and transverse to $\mathcal{D}|_{K_1 \times S^1}$.*

Proof. The idea is the same as for Lemma 3.12. Consider a base \mathcal{B}_{K_2} of the cone of \mathcal{D} -structure currents in $K_2 \times S^1$. Further consider the subspace of structure cycles \mathcal{Z}_{K_2} in the kernel of $d\theta$. The closed space spanned by \mathcal{Z}_{K_2} and the kernel of ω_1 is disjoint from \mathcal{B}_{K_2} . Since \mathcal{B}_{K_2} is compact and convex the Theorem of Hahn-Banach allows to extend the subspace to a closed hyperplane \mathcal{H} of codimension one and disjoint from \mathcal{B}_{K_2} . Setting $\omega_2|_{K_1 \times S^1} \equiv \omega_1$ and $\ker \omega_2 = \mathcal{H}$ yields the extension. \square

Proof of Theorem 2.3. Choose a compact exhaustion of M . Taking the direct limit of the forms in Lemma 3.16 yields a closed form ω transverse to \mathcal{D} which lifts to an exact form in the covering $p: M \times \mathbb{R} \rightarrow M \times S^1$. Choose a primitive T of $p^*\omega$. The image of the projection onto M of every levelset of T is M itself since it is open and closed in M . This is due to the fact that there is a local bound on the angle of $\ker \omega$ with TM . By construction every levelset is the graph of a function $-\tau: M \rightarrow \mathbb{R}$ with $-d\tau|_C \leq L$. \square

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